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Visibility of ideal classes

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ABSTRACT

Cremona, Mazur, and others have studied what they call visibility of elements of Shafarevich–Tate groups of elliptic curves. The analogue for an abelian number field K is capitulation of ideal classes of K in the minimal cyclotomic field containing K. We develop a new method to study capitulation and use it and classical methods to compute data with the hope of gaining insight into the elliptic curve case. For example, the numerical data for number fields suggests that visibility of non-trivial Shafarevich–Tate elements might be much more common for elliptic curves of positive rank than for curves of rank 0.

Let E be an elliptic curve over $\mathbf{Q}$ of conductor N. Then there is a modular parametrization $X_0(N) \to E$ and a corresponding map $E \to J_0(N)$ of E into the jacobian of $X_0(N)$. This induces a map of Shafarevich–Tate groups $Ш(E) \to Ш(J_0(N))$. Cremona and Mazur [6] study the question of when an element s of $Ш(E)$ is in the kernel of this map. When this happens, there is a curve defined over $\mathbf{Q}$ and contained in $J_0(N)$ that represents s. In other words, s is "visible" in $J_0(N)$. Further work on this topic has been done by Agashe, Stein, and others [1,2].

Let $K/\mathbf{Q}$ be an abelian extension of conductor n, so $K \subseteq \mathbf{Q}(\zeta_n)$, where n is the conductor of K. This is the analogue of the modular parametrization above. The ideal class group is the analogue of the Shafarevich–Tate group (this will be made more precise in Section 1), so the analogue of the above question is to ask when ideal classes of K become principal in $\mathbf{Q}(\zeta_n)$.

Let L/K be an extension of number fields. An ideal class of K that becomes principal in L is said to *capitulate*. Many authors have discussed capitulation in various contexts. The hope of the present

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paper is to use results about capitulation for $\mathbf{Q}(\zeta_n)/K$ to gain some insight into the situation for Shafarevich–Tate groups.

For example, for imaginary quadratic fields K, the capitulation in $\mathbf{Q}(\zeta_n)/K$ is mostly accounted for by the ambiguous classes, namely those produced by genus theory, and such classes are easily seen to capitulate. In examples of small discriminant, the class group consists mostly of ambiguous classes, so capitulation is very common. However, for large discriminants, most of the ideal class group does not capitulate (see the discussion after Proposition 4).

The situation for real quadratic fields is quite different. The ambiguous classes capitulate, but numerical data indicates that capitulation of additional classes is very common. The situation for cyclic cubic fields is similar.

Let's return to Shafarevich–Tate groups. Cremona and Mazur looked at elliptic curves of conductor up to 5500. All of their examples of non-trivial $Ш(E)$ had Mordell–Weil rank 0, and capitulation (i.e., visibility) was very common. The analogue of the Mordell–Weil group is the unit group of the ring of integers of a number field. An elliptic curve with Mordell–Weil rank 0 therefore corresponds to an imaginary quadratic field (or $\mathbf{Q}$). As pointed out above, capitulation is very common for imaginary quadratic fields with small discriminants. The numerical results of Cremona and Mazur for elliptic curves match this situation. By analogy, it is natural to ask the following questions. Should we expect non-visibility to become the rule rather than the exception as the conductor increases? Is there an analogue of genus theory that accounts for most of the Shafarevich–Tate group for small conductors? We do not know the answer to either question. Both seem worth investigating. In Section 2, we show that the visibility of certain elements of Shafarevich–Tate groups, as proved by Cremona and Mazur, is analogous to the capitulation of ideals of quadratic fields in biquadratic fields. But the Cremona–Mazur method does not produce non-trivial elements of the Shafarevich–Tate group; it shows only that they are visible, in contrast to the situation in genus theory for ideal class groups.

Real quadratic fields (along with non-real cubics and totally complex quartics) correspond to elliptic curves with Mordell–Weil rank 1. Cyclic cubic fields correspond to elliptic curves of rank 2. The results for real quadratic fields and cyclic cubic fields suggest that, for elliptic curves of positive rank, visibility should be very common. There is very little data available; again, this seems to be worth investigating.

The Cohen–Lenstra heuristics [5] predict that, for totally real fields, the existence of units makes class numbers tend to be small. Therefore, it is perhaps common that the class number of a totally real abelian number field has approximately the same class number as the minimal real cyclotomic field containing it (this is hard to check numerically, since class numbers of real cyclotomic fields are hard to calculate; but see [18] for some probable examples). This tends to force capitulation of ideal classes (see part (i) of Lemma 4 in Section 4). Is there is an analogous situation for elliptic curves of positive rank? Of course, there is a difference in this case between the number field case and the elliptic curve case. When a number field is properly contained in a cyclotomic field, the rank of the group of units of the number field is less than that of the cyclotomic field (except when the number field is the real subfield of the cyclotomic field). There does not seem to be a reason for an analogous situation for the Mordell–Weil ranks of an elliptic curve and the corresponding $J_0(N)$.

1. The analogy between ideal class groups and Shafarevich–Tate groups

In this section we review and make explicit some analogies between ideal class groups and Shafarevich–Tate groups.

Let E be an elliptic curve defined over $\mathbf{Q}$. The Shafarevich–Tate group is defined to be the group of everywhere locally trivial elements of $H^1(G_{\mathbf{Q}}, E(\overline{\mathbf{Q}}))$, namely

$$Ш(E) = Ker\left(H^1\big(G_{\mathbf{Q}}, E(\overline{\mathbf{Q}})\big) \to \prod_{p\leqslant\infty} H^1\big(G_{\mathbf{Q}_p}, E(\overline{\mathbf{Q}}_p)\big)\right).$$

Here $G_L = \mathrm{Gal}(\overline{L}/L)$ for any field L, and embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ are implicitly fixed.

Now let's consider the number field case. Let K be a number field. Since units are the analogue of points on an elliptic curve, we let E denote the group of all units in the ring of integers of $\overline{\mathbf{Q}}$. For a place $\mathfrak{p}$ of K, let $K_\mathfrak{p}$ be the completion at $\mathfrak{p}$. When $\mathfrak{p}$ is finite, let p denote the rational prime below $\mathfrak{p}$ and let U_p be the group of units in the integral closure of $\mathbf{Z}_p$ in $\overline{\mathbf{Q}}_p$. When $\mathfrak{p}$ is archimedean, let $U_p = \mathbf{C}^\times$. Let C_K denote the ideal class group of K. The following result has been implicit in the literature (for example, apply the techniques of [15, pp. 98–99] to the inclusion j: Spec $K \to$ Spec O_K for the étale sheaf G_m) and also appears in [7] and [9, Lemma 6.1]. Since we need it and its proof later, we include it. It shows the strong analogy between ideal class groups and Shafarevich–Tate groups. The cohomology groups are the standard profinite cohomology groups defined using continuous cocycles.

Proposition 1. *There is an isomorphism*

$$C_K \simeq Ker\left(H^1(G_K, E) \to \prod_{\mathfrak{p}} H^1(G_{K_\mathfrak{p}}, U_p) \right).$$

Proof. The product is over the places of K. Since $H^1(G_\mathbf{R}, \mathbf{C}^\times) = 0$ and $H^1(G_\mathbf{C}, \mathbf{C}^\times) = 0$, the archimedean primes can be ignored in the statement of the theorem and in the following.

Let I be an ideal of K. Then I becomes principal in some extension of K, so $I = (\beta)$ for some $\beta \in \overline{\mathbf{Q}}$ (we use I to denote the lift of I to extension fields). Define $c_I : \mathrm{Gal}(\overline{\mathbf{Q}}/K) \to E$ by $\sigma \mapsto \beta^{\sigma-1}$. Since $\sigma(I) = I$, we have $\beta^{\sigma-1} \in E$. It follows easily that c_I is a continuous cocycle. If also $I = (\beta_1)$, then $\epsilon = \beta/\beta_1 \in E$, so the cocycle defined using β_1 differs from c_I by the coboundary $\sigma \mapsto \epsilon^{\sigma-1}$. Therefore the cohomology class of c_I depends only on I. In fact, c_I depends only on the ideal class of I: if $a \in K^\times$ then

$$c_{aI}(\sigma) = (a\beta)^{\sigma-1} = \beta^{\sigma-1} = c_I(\sigma).$$

Therefore we have a homomorphism $\phi : C_K \to H^1(G_K, E)$.

Suppose c_I is a coboundary, which means there is some $\epsilon \in E$ such that $\beta^{\sigma-1} = \epsilon^{\sigma-1}$ for all σ. This means that $\beta/\epsilon \in K$, so $I = (\beta/\epsilon)$ is principal in K. Therefore ϕ is injective.

We now show that c_I is locally trivial. Fix a prime ideal $\mathfrak{p}$ of K. Choose an ideal J in the ideal class of I with J prime to $\mathfrak{p}$. Then $J = (\gamma)$ for some $\gamma \in \overline{\mathbf{Q}}$, and $\gamma \in U_p$ via the embedding of $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ induced by $\mathfrak{p}$. Therefore c_J restricted to $G_{K_\mathfrak{p}}$ is given by the coboundary $\sigma \mapsto \gamma^{\sigma-1}$, so the cohomology class of $c_I = c_J$ is locally trivial.

Finally, we show ϕ is surjective. Let c be a cocycle in $H^1(G_K, E)$ that is locally a coboundary. Hilbert's Theorem 90 says that $H^1(G_K, \overline{\mathbf{Q}}^\times) = 0$. The map $E \to \overline{\mathbf{Q}}^\times$ therefore sends c to a coboundary, so $c(\sigma) = y^{\sigma-1}$ for some $y \in \overline{\mathbf{Q}}^\times$. Since c is continuous, c has finite order, hence c^m is a coboundary for some m. Therefore, $c^m(\sigma) = \epsilon^{\sigma-1}$ for some $\epsilon \in E$. This implies that y^m/ϵ is fixed by all σ, hence is in K. Let $\alpha = y^m/\epsilon$.

Let $\mathfrak{p}$ be a prime ideal of K. Since c is a coboundary at $\mathfrak{p}$, we have $c(\sigma) = u_\mathfrak{p}^{\sigma-1}$ for all $\sigma \in G_{K_\mathfrak{p}}$ for some $u_\mathfrak{p} \in U_p$. Therefore, $y^{\sigma-1} = u_\mathfrak{p}^{\sigma-1}$ for all $\sigma \in G_{K_\mathfrak{p}}$, hence $y/u_\mathfrak{p} \in K_\mathfrak{p}$. Therefore

$$v_\mathfrak{p}(\alpha) = v_\mathfrak{p}\big(y^m\big) = v_\mathfrak{p}\big((y/u_\mathfrak{p})^m\big) \equiv 0 \pmod{m}.$$

Since this happens for all $\mathfrak{p}$, we must have $(\alpha) = I^m$ for some ideal I of K.

Let $\epsilon_1 = \epsilon^{1/m} \in E$. Then $I = (y/\epsilon_1)$ in some extension of K. It follows easily that the cohomology class of c_I equals the cohomology class of c. Therefore ϕ is surjective. □

Let L/K be a finite Galois extension of number fields and let E_L be the units of the ring of integers of L. Define the locally trivial cohomology group to be

$$H^1_{lt}(L/K, E_L) = Ker\left(H^1\big(\mathrm{Gal}(L/K), E_L\big) \to \prod_{\mathfrak{p}} H^1\big(\mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}}), U_{L_{\mathfrak{p}}}\big)\right),$$

where $L_{\mathfrak{p}}$ denotes the completion of L at one of the primes of L above $\mathfrak{p}$ and $U_{L_{\mathfrak{p}}}$ is the group of local units in $L_{\mathfrak{p}}$.

The inclusion map $K \hookrightarrow L$ induces a map $C_K \to C_L$. The following result appears in [16].

Corollary. *There is an isomorphism*

$$Ker(C_K \to C_L) \simeq H^1_{lt}(L/K, E_L).$$

Proof. The beginning of the inflation–restriction exact sequence is

$$0 \to H^1(L/K, E_L) \to H^1(G_K, E) \to H^1(G_L, E).$$

An element $x \in H^1_{lt}(L/K, E_L)$ clearly yields an element of $H^1(G_K, E)$ that is locally trivial, hence corresponds to an element $y \in C_K$. The map from $H^1(G_K, E)$ to $H^1(G_L, E)$, when restricted to locally trivial elements, is easily seen to correspond to the map on class groups. Since x is 0 in $H^1(G_L, E)$, it follows that y is 0 in C_L. Therefore we have a map $\psi : H^1_{lt}(L/K, E_L) \to Ker(C_K \to C_L)$. The injectivity of ψ is immediate from the injectivity on the left in the inflation–restriction sequence.

It remains to show that ψ is surjective. An element of $Ker(C_K \to C_L)$ corresponds to a cocycle c whose class in $H^1(L/K, E_L)$ is locally trivial when regarded as an element $\tilde{c} \in H^1(G_K, E)$. We must show that c is locally trivial in $H^1(L/K, E_L)$. Since $\tilde{c}$ is the inflation of c, we have $\tilde{c}(\sigma) = 1$ for all $\sigma \in G_L$. Let $\mathfrak{p}$ be a prime ideal of K. The local triviality in $H^1(G_K, E)$ implies that there exists $u_{\mathfrak{p}} \in U_{\mathfrak{p}}$ such that $\tilde{c}(\sigma) = u_{\mathfrak{p}}^{\sigma-1}$ for all $\sigma \in G_{K_{\mathfrak{p}}}$. Since $u_{\mathfrak{p}}^{\sigma-1} = \tilde{c}(\sigma) = 1$ for all $\sigma \in G_L \cap G_{K_{\mathfrak{p}}} = G_{L_{\mathfrak{p}}}$, we have $u_{\mathfrak{p}} \in U_{L_{\mathfrak{p}}}$. This means that $c \in H^1_{lt}(L/K, E_L)$. Therefore ψ is surjective. □

In the case of elliptic curves, the fundamental descent sequence is

$$0 \to E(\mathbf{Q})/nE(\mathbf{Q}) \to S_n \to Ш[n] \to 0,$$

where $S_n \subseteq H^1(G_{\mathbf{Q}}, E[n])$ is the n-Selmer group. There is an analogue for number fields. Recall that, for a number field K,

$$H^1(G_K, \mu_n) \simeq K^\times/\big(K^\times\big)^n.$$

This follows easily from Hilbert's Theorem 90.

Proposition 2. *Let K be a number field and let $n \geqslant 1$. Let*

$$S_n = \big\{x \in K^\times \;\big|\; (x) = I^n \text{ for some ideal } I \subset K\big\}/\big(K^\times\big)^n.$$

Then there is an exact sequence

$$1 \to E_K/(E_K)^n \to S_n \to C_K[n] \to 1,$$

where $C_K[n]$ denotes the n-torsion in C_K and where the map from S_n to $C_K[n]$ sends x to the ideal class of I. Also, S_n is the inverse image of $H^1_{lt}(G_K, E_K)[n]$ under the maps

$$K^\times/\big(K^\times\big)^n \simeq H^1(G_K, \mu_n) \to H^1(G_K, E_K).$$

Proof. The exactness of the sequence is straightforward. To verify the last claim, use the fact that $g \mapsto g(x^{1/n})/x^{1/n}$ gives the cocycle in $H^1(G_K, \mu_n)$ corresponding to x. Moreover, $I = (x^{1/n})$ in $K(x^{1/n})$, so this is also the cocycle in $H^1_{lt}(G_K, E)$ corresponding to the ideal class of I under the isomorphism of Proposition 1. □

Remark. In the elliptic curve situation, we are interested in

$$Ker\big(Ш(E) \to Ш\big(J_0(N)\big)\big) \subseteq Ker\big(H^1\big(G_{\mathbf{Q}}, E(\overline{\mathbf{Q}})\big) \to H^1\big(G_{\mathbf{Q}}, J_0(N)(\overline{\mathbf{Q}})\big)\big).$$

In the number field case, we have

$$Ker(C_K \to C_L) \hookrightarrow Ker\big(H^1(G_K, E) \to H^1(G_L, E)\big).$$

This map is obtained from the map on Galois groups rather than a map on E, which would be closer to the geometric situation. However, this can easily be remedied. Let

$$J_L = \mathrm{Hom}_{G_L}\big(\mathbf{Z}[G_{\mathbf{Q}}], E\big) = \mathrm{Maps}(G_{\mathbf{Q}}/G_L, E).$$

Shapiro's Lemma says that $H^1(G_{\mathbf{Q}}, J_L) \simeq H^1(G_L, E)$. Therefore, we have

$$Ker(C_K \to C_L) \hookrightarrow Ker\big(H^1(G_{\mathbf{Q}}, J_K) \to H^1(G_{\mathbf{Q}}, J_L)\big),$$

and the map is obtained from the natural map $J_K \hookrightarrow J_L$.

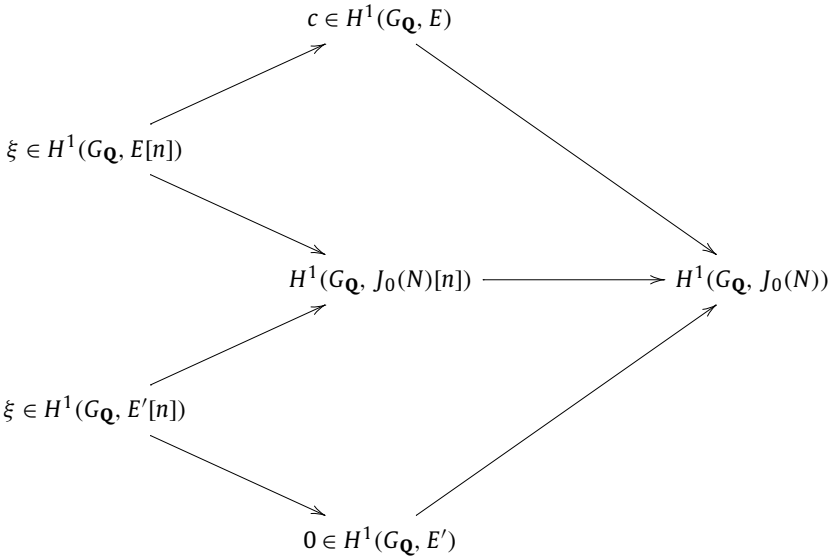
Remark. It is known (see [6]) that $Ш(E)$ becomes trivial in some abelian variety containing E. The question has been raised whether there are classes of $Ш(E)$ that are not visible in $J_0(N)$ but which become trivial in $J_0(M)$ for some M that is a multiple of N. The analogous question can be asked for number fields. The following example shows that this situation can arise. Consider the cubic subfield K of $\mathbf{Q}(\zeta_{163})$. It was shown by Kummer that the class group of K is the product of two groups of order 2. Since $[\mathbf{Q}(\zeta_{163})^+ : K] = 27$, and since the map on ideal classes from the real subfield to the full cyclotomic field is injective, these classes do not capitulate in $\mathbf{Q}(\zeta_{163})$. In [10], G. Gras points out that the ideal class group of K capitulates in $K(\sqrt{13})$. Since $\sqrt{13} \in \mathbf{Q}(\zeta_{13})$, the ideal class group of K capitulates in $\mathbf{Q}(\zeta_{13\cdot 163})$.

In contrast to this example, the result of Brumer [4] given in Section 3 implies that there are ideal classes in some cyclotomic fields that do not become principal in any cyclotomic field.

2. Creating visible elements

One of the methods Cremona and Mazur use for identifying visible elements of Shafarevich–Tate groups is the following (see [6, p. 19]). Let E and E' be elliptic curves contained in $J_0(N)$

and assume $E[n] = E'[n]$ as subgroups of $J_0(N)$. Suppose $c \in Ш(E) \subseteq H^1(G_{\mathbf{Q}}, E(\overline{\mathbf{Q}}))$ is the image of $\xi \in H^1(G_{\mathbf{Q}}, E[n])$. If ξ maps to $0 \in H^1(G_{\mathbf{Q}}, E'(\overline{\mathbf{Q}}))$, then c maps to $0 \in Ш(J_0(N))$, hence is visible.



In this situation, there exists $R \in E'(\overline{\mathbf{Q}})$ such that $\sigma(R) - R = \xi(\sigma)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Since $\xi(\sigma) \in E'[n]$, we have $\sigma(nR) = nR$ for all σ, so $nR \in E'(\mathbf{Q})$.

When $n = 2$, this situation is very much analogous to the fact that ideal classes of order 2 in quadratic fields capitulate in suitably chosen biquadratic fields. Let $K = \mathbf{Q}(\sqrt{d})$ be a quadratic field of discriminant d and let $1 < d_1 < |d|$ be a fundamental discriminant dividing d (if such a d_1 exists). Let $F = \mathbf{Q}(\sqrt{d_1})$ and $L = K(\sqrt{d_1})$. Then L/K is an unramified quadratic extension. Moreover, L is contained in the smallest cyclotomic field containing K.

The number $d_1 \in K^\times/(K^\times)^2 \simeq H^1(K, \mu_2)$ yields an ideal J of K with $J^2 = (d_1)$. Since d_1 represents the trivial class in $F^\times/(F^\times)^2$, it also represents the trivial class in $L^\times/(L^\times)^2$, so we recover the obvious fact that J becomes principal in L.

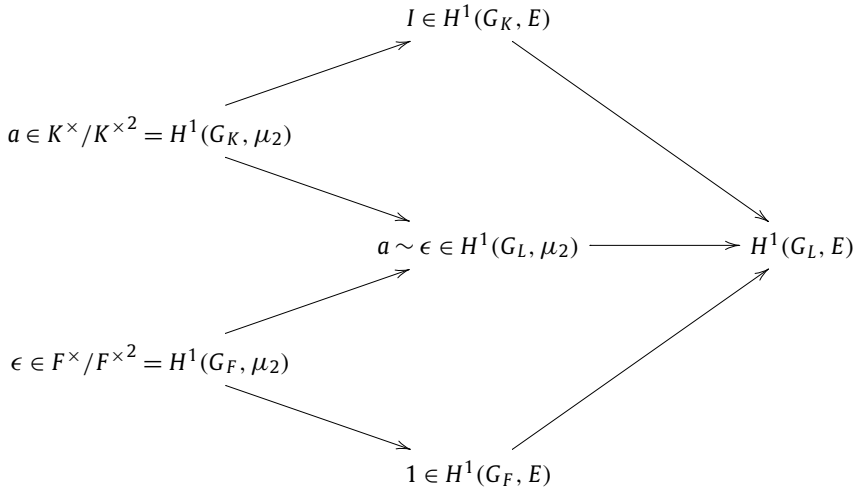
A more interesting example is obtained as follows. Let ϵ be the fundamental unit of F. Then, in the notation of Proposition 2, ϵ represents a non-trivial element of $S_2(F)$ that has trivial image in C_F. Assume that the norm of ϵ is $+1$. Let $\alpha = 1 + \epsilon^{-1}$. The norm of α is $a = (1 + \epsilon^{-1})(1 + \epsilon) \in \mathbf{Z}$. Let σ be the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q})$. Then $\alpha^\sigma/\alpha = \epsilon$. Therefore the ideal (α) of F is fixed by $\mathrm{Gal}(F/\mathbf{Q})$, and therefore also by $\mathrm{Gal}(L/K)$. Since L/K is unramified, there is an ideal I of K such that $I = (\alpha)$ in L. Moreover, $\alpha^2\epsilon = \alpha\alpha^\sigma = a \in \mathbf{Z}$, which implies that $I^2 = (a)$ in K. The coset of

$$a = \alpha^2\epsilon \in K^\times/\left(K^\times\right)^2 \simeq H^1(G_K, \mu_2)$$

maps to the coset of

$$\epsilon \in L^\times/\left(L^\times\right)^2 \simeq H^1(G_L, \mu_2).$$

This coset maps to the trivial ideal class in C_L, corresponding to the fact that the ideal class of I capitulates in L. This is clearly an analogue of the elliptic curve situation described above, where $\sqrt{\epsilon}$ corresponds to the point R.



It is natural to ask about the class of the ideal I just constructed. It can be identified via the following result.

Lemma 1. *Let $\epsilon = (x + y\sqrt{d_1})/2$ be the fundamental unit of $\mathbf{Q}(\sqrt{d_1})$. Then $x + 2 = rw^2$ for some positive integers r, w such that $r|2d_1$ and such that r and $4d_1/r$ are not squares.*

Proof. We have $x^2 - d_1y^2 = 4$, so $(x + 2)(x - 2) = d_1y^2$. If $x + 2$ or $x - 2$ is a square, then $\sqrt{\epsilon} = \frac{1}{2}(\sqrt{x+2} + \sqrt{x-2}) \in \mathbf{Q}(\sqrt{d_1})$, which is a contradiction. Since $\gcd(x + 2, x - 2)$ divides 4, the result follows easily. □

Since $I^2 = (a)$ with $a = (1 + \epsilon^{-1})(1 + \epsilon) = 2 + x = rw^2$ in the notation of the lemma, we find that $(I/w)^2 = (r)$. Therefore the ideal class of I comes from the class of r in $K^\times/(K^\times)^2$. The ideal class of I is non-trivial in K but capitulates in L. This capitulation represents "non-obvious" capitulation in L (where the "obvious" capitulation is for the ideal whose square is (d_1)). Moreover, the factorization of $(1 + \epsilon)$, which equals I/w in L, into a product of primes gives the "non-obvious" relation in the class group of F (where the "obvious" relation is that the product of all ramified primes, with a possible omission of the prime above 2, is principal).

3. Capitulation of ideal classes: General results

Lemma 2. *Let $K \subseteq L$ be number fields with $[L : K] = n$. Let I be an ideal of K. If I becomes principal in L then I^n is principal in K.*

Proof. If I is principal in L, its norm is principal in K. But the norm is I^n. □

Remark. The analogue of this result is true for elliptic curves: If an element of Ш(E) is visible, then its order divides the degree of modular parametrization of E (see [6]).

Let $K/\mathbf{Q}$ be an abelian extension of degree d and of conductor n, so $K \subseteq \mathbf{Q}(\zeta_n)$. We say that an ideal class has *potential capitulation* if its order divides $\phi(n)/d$. We say that K has *maximal capitulation*

if all ideal classes with potential capitulation actually capitulate in $\mathbf{Q}(\zeta_n)$, and *maximal p-capitulation* if all classes of p-power order with potential capitulation actually capitulate in $\mathbf{Q}(\zeta_n)$.

The relevant question to consider is whether classes with potential capitulation actually capitulate. There is a marked difference in the behaviors of the real and imaginary ideal classes. The real case is related to the result of Kurihara [12] (see also [10]) that says that if K is totally real, then all ideal classes of K capitulate in the field $K(\zeta_\infty)$ obtained by adjoining all roots of unity to K. On the other hand, a result of Brumer [4] says that the class group (defined as a direct limit) of the extension of $\mathbf{Q}$ generated by all roots of unity is isomorphic to a countable direct sum of factors $\mathbf{Q}/\mathbf{Z}$. By Kurihara's result, these classes cannot arise from class groups of real fields. Of course, both of these results relate to capitulation in fields much larger than the minimal cyclotomic field containing K. But they give an indication of the difference between the two cases.

There is an explanation of why there should be a lot of capitulation in the real case. The Cohen–Lenstra heuristics [5] predict that class numbers of real fields tend to be small. Suppose a prime p divides the class number of $K \subseteq \mathbf{Q}(\zeta_n)^+$. Since h_K tends to be small, it is likely that $p^2 \nmid h_K$. Now suppose that p divides the class number h_n^+ of the real cyclotomic field. By the same reasoning, it is likely that $p^2 \nmid h_n^+$. By Lemma 4(i) below, the classes of order p in K capitulate in this case. We shall see examples of this phenomenon in Section 6.

The following result is useful when working with totally real fields K of prime conductor ℓ. It shows that we need to consider capitulation only from K to the real subfield $\mathbf{Q}(\zeta_\ell)^+$ of the cyclotomic field.

Proposition 3. *Let ℓ be prime. Then the map from the class group of $\mathbf{Q}(\zeta_\ell)^+$ to the class group of $\mathbf{Q}(\zeta_\ell)$ is injective.*

Proof. See [19, Theorem 4.14]. □

4. Classical methods

To treat the imaginary classes, we need the following.

Lemma 3. *Let K be a number field contained in the nth cyclotomic field $\mathbf{Q}(\zeta_n)$, and let d be the number of roots of unity in K. Let μ_m be the group of roots of unity in $\mathbf{Q}(\zeta_n)$ (so $m = n$ or $2n$). Then $H^1(G_{\mathbf{Q}(\zeta_n)/K}, \mu_m)$ is annihilated by d.*

Proof. Let G be a group, let $t \in G$, and let A be a G-module. The automorphism $g \mapsto tgt^{-1}$ gives A a new module structure; call it A^t. The map $\psi : a \mapsto t^{-1}a$ is a G-homomorphism from A^t to A. Proposition 3 of [3] says that the composite map

$$H^1(G, A) \to H^1\big(G, A^t\big) \xrightarrow{\psi_*} H^1(G, A)$$

is the identity map.

In our case, identify $\mathrm{Gal}(\mathbf{Q}(\zeta_n)/K)$ with a subgroup G of $(\mathbf{Z}/m\mathbf{Z})^\times$. The module $A = \mu_m$ becomes $\mathbf{Z}/m\mathbf{Z}$ with G acting by multiplication. Since conjugation by $t \in G$ is trivial, $A^t = A$. The map ψ is multiplication by an integer $t' \equiv t^{-1} \pmod m$, so the map on cohomology is also multiplication by t'. Therefore

$$t' : H^1(G_{\mathbf{Q}(\zeta_n)/K}, \mu_m) \to H^1(G_{\mathbf{Q}(\zeta_n)/K}, \mu_m)$$

is the identity for all integers $t' \in G$. Let $d = \gcd(\{t' - 1\}, m)$, where t' runs through all such integers. Then d annihilates $H^1(G_{\mathbf{Q}(\zeta_n)/K}, \mu_m)$.

If $\zeta \in \mu_m$, then $\zeta \in K$ if and only if $\zeta^{t'} = \zeta$ for all $t' \in G$. Therefore $\zeta \in K$ if and only if $\zeta^d = 1$. This means that there are exactly d roots of unity in K. □

Proposition 4. *Let K be a subfield of $\mathbf{Q}(\zeta_n)$ and let d be the number of roots of unity in K. Let I be an ideal of K that becomes principal in $\mathbf{Q}(\zeta_n)$. Then $(I/\bar{I})^d$ is principal in K (where $\bar{I}$ denotes the complex conjugate).*

Proof. Suppose $I = (\alpha)$ in $\mathbf{Q}(\zeta_n)$. Let $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_n)/K)$. Then $\alpha^{\sigma-1} \in E_{\mathbf{Q}(\zeta_n)}$. Therefore, $(\alpha/\overline{\alpha})^{\sigma-1}$ is a unit of absolute value 1, hence a root of unity. It follows that the map $\sigma \mapsto (\alpha/\overline{\alpha})^{\sigma-1}$ is a cocycle for $H^1(G_{\mathbf{Q}(\zeta_n)/K}, \mu_m)$, where μ_m is the group of all roots of unity in $\mathbf{Q}(\zeta_n)$. By Lemma 3, there exists $\zeta \in \mu_m$ such that $(\alpha/\overline{\alpha})^{d(\sigma-1)} = \zeta^{\sigma-1}$ for all σ. This implies that $(\alpha/\overline{\alpha})^d/\zeta \in K$, hence that $(I/\bar{I})^d$ is principal in K. □

Corollary. *Let K be a subfield of $\mathbf{Q}(\zeta_n)$ and suppose that the only roots of unity in K are ± 1. If I is an ideal of K such that $I\bar{I}$ is principal in K and I is principal in $\mathbf{Q}(\zeta_n)$, then I^4 is principal in K.*

The corollary applies in particular to imaginary quadratic fields. In this case, the 4-torsion of the class group is of order at most 4^{s-1}, where s is the number of prime factors of the discriminant. It is therefore an easy consequence of Siegel's theorem (that $\log(h) \sim \frac{1}{2}\log(|d|)$) that the 4-torsion is only a small part of the class group when the discriminant is large, and therefore most of the class group does not capitulate in $\mathbf{Q}(\zeta_d)$.

Example. Here is an example where a class of order 4 capitulates. Let $K = \mathbf{Q}(\sqrt{-39})$, whose class group is cyclic of order 4, generated by the ideal $I = (2, \frac{1+\sqrt{-39}}{2})$. The ideal $J = (3, \frac{-3+\sqrt{-39}}{2})$ is not principal but satisfies $J^2 = (3)$, hence has order 2 in the class group of K. Therefore I^2 is in the same class as J. Since J becomes principal in $L = \mathbf{Q}(\sqrt{-39}, \sqrt{-3})$, namely $J = (\sqrt{-3})$, it follows that I^2 is principal in L. However, I cannot be principal in L since then the norm from L to K of I, namely I^2, would be principal in K, which is not the case. The class number of $\mathbf{Q}(\zeta_{39})$ is 2. Since $\mathbf{Q}(\zeta_{39})/L$ is totally ramified, the norm $N : C_{\mathbf{Q}(\zeta_{39})} \to C_L$ is surjective. If I is not principal in $\mathbf{Q}(\zeta_{39})$, then it generates the class group, hence $N(I) = I^6$ generates the class group of L; contradiction. Therefore I is principal in $\mathbf{Q}(\zeta_{39})$.

For a quadratic field, the classes of order 2 always capitulate in the cyclotomic field. This is easily seen as follows: Let $K = \mathbf{Q}(\sqrt{d})$, where d is the discriminant of K. The ideal classes of order 2 are generated by ideals I with $I^2 = (r)$ and r dividing d. However, $\mathbf{Q}(\zeta_{|d|})$ is the smallest cyclotomic field containing K, and $\mathbf{Q}(\zeta_{|d|})$ contains $\sqrt{\pm r}$ for each such r and an appropriate choice of sign. Therefore each I becomes principal in $\mathbf{Q}(\zeta_{|d|})$.

More generally, Furuya [8] has shown that when $K/\mathbf{Q}$ is an abelian extension, every ideal fixed by $\mathrm{Gal}(K/\mathbf{Q})$ becomes principal in the genus field of K (i.e., the maximal abelian extension of $\mathbf{Q}$ that is unramified over K). Since the genus field is contained in the smallest cyclotomic field containing K, all such ideals capitulate in the cyclotomic field.

The following result is useful in many cases.

Lemma 4. *Suppose ℓ is prime and L/F is an extension of number fields. Also, suppose L/F has no non-trivial unramified subextensions M/F. Let h_F and h_L be the class numbers of F and L.*

(i) *Assume $[L : F] = \ell^a$. If $\ell \nmid h_L/h_F$, then the kernel of the map $C_F \to C_L$ is exactly the classes of order dividing ℓ^a.*
(ii) *Assume ℓ is odd and L/F is Galois of degree ℓ. If the ℓ-power part of the class group of L is cyclic of order ℓ^k and the ℓ-power part of the class group of F has order ℓ^f with $f < k$, then $f = k - 1$ and the map $C_F \to C_L$ is injective.*
(iii) *Assume ℓ is odd and L/F is Galois of degree ℓ. If the ℓ-power part of the class group of L is isomorphic to $\mathbf{Z}/\ell\mathbf{Z} \times \mathbf{Z}/\ell\mathbf{Z}$, then all classes of order ℓ in F become principal in L.*

Proof. (i) The only possible capitulation occurs in the ℓ^a-torsion. Let A_F and A_L be the ℓ-power parts of the class groups. Since the norm map from A_L to A_F is surjective, it must be an isomorphism since

the groups have the same order. Let I represent a class in A_F of order dividing ℓ^a. Lifting I to L then taking the norm back to F yields I^{ℓ^a}, which is principal. Since the norm is injective, the image of I in C_L must have been trivial.

(ii) The case $k = 1$ is trivial, so assume $k > 1$. The map $C_F \to C_L$ is injective on the non-ℓ parts, so we restrict our attention to A_F and A_L. By assumption, $A_L \simeq \mathbf{Z}/\ell^k\mathbf{Z}$. A generator σ of $\mathrm{Gal}(L/F)$ acts on $\mathbf{Z}/\ell^k\mathbf{Z}$ as an automorphism of order 1 or ℓ, hence by multiplication by $1 + a\ell^{k-1}$ for some a. It follows easily that $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{\ell-1}$ acts as multiplication by ℓ. But N is the composition of the two maps

$$A_L \to A_F \to A_L,$$

where the first map is the norm and the second is the natural map on class groups. Since the image of N has order ℓ^{k-1} and A_F is assumed to have order at most this large, A_F has order exactly ℓ^{k-1}. Also, $A_F \to A_L$ must be an injection.

(iii) Let σ generate $\mathrm{Gal}(L/F)$. Then σ is a linear transformation of the ℓ-part of the class group of L, which is a $\mathbf{Z}/\ell\mathbf{Z}$ vector space. The Jordan canonical form of σ is $M = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some b. Therefore,

$$1 + \sigma + \cdots + \sigma^{\ell-1} = I + M + \cdots + M^{\ell-1} \equiv 0 \pmod{\ell},$$

so $1 + \sigma + \cdots + \sigma^{\ell-1}$ annihilates the classes of L of order ℓ. But $1 + \sigma + \cdots + \sigma^{\ell-1}$ is the norm map followed by the natural map from the class group of F to the class group of L. Since L/F is totally ramified, the norm map is surjective. Therefore, all classes of order ℓ in F must become principal in L. □

5. Galois module methods

In this section, we introduce a method based on [18] that is much more powerful than those of the previous section when working with totally real abelian fields of prime conductor.

Let p be prime, let $\ell \equiv 1 \pmod{p}$ also be prime, and let $G = \mathrm{Gal}(\mathbf{Q}(\zeta_\ell)^+/\mathbf{Q})$. Let π be the maximal subgroup of G of p-power order and Δ the maximal subgroup of G of order prime to p. Then $G = \pi \times \Delta$. Let p^n be the order of π.

Let $\chi : \Delta \to \overline{\mathbf{Q}}_p^\times$ be a p-adic valued Dirichlet character and let $\mathcal{O}_\chi$ be the extension of $\mathbf{Z}_p$ generated by the values of χ. It is a $\mathbf{Z}[\Delta]$-algebra via $\delta \cdot x = \chi(\delta)x$ for $\delta \in \Delta$ and $x \in \mathcal{O}_\chi$. For any $\mathbf{Z}[G]$-module M, define its χ-eigenspace to be

$$M(\chi) = M \otimes_{\mathbf{Z}[\Delta]} \mathcal{O}_\chi.$$

The functor $M \mapsto M(\chi)$ is exact. We have that

$$M \simeq \prod_\chi M(\chi),$$

where χ runs through representatives for the $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-conjugacy classes of characters of Δ. The eigenspace $M(\chi)$ has the natural structure of an $\mathcal{O}_\chi[\pi]$-module. Moreover,

$$\mathcal{O}_\chi[\pi] \simeq \mathcal{O}_\chi[[T]]/\big(\omega_n(T)\big),$$

where $1 + T$ corresponds to the choice of a generator of π and $\omega_n(T) = (1+T)^{p^n} - 1$.

Let F be the fixed field of π. So Δ is identified with $\mathrm{Gal}(F/\mathbf{Q})$. Let $H \subseteq \Delta$ be the kernel of χ and let $K \subseteq F$ be the fixed field of H.

Let A_K denote the p-Sylow subgroup of the class group of K, and similarly for other fields.

Lemma 5. *The natural map $A_K \to A_F$ yields an isomorphism $A_K(\chi) \simeq A_F(\chi)$.*

Proof. Let $N : A_F(\chi) \to A_K(\chi)$ be the norm map. The natural map $A_K(\chi) \to A_F(\chi)$ followed by N is the $|H|$-th power map. Since $p \nmid |H|$, this is an injection, so $A_K(\chi) \to A_F(\chi)$ is injective. Since $\chi(H) = 1$, the map $A_F(\chi) \to A_F(\chi)$ given by N followed by the natural map from $A_K(\chi)$ to $A_F(\chi)$ is the $|H|$-th power map, so $A_K(\chi) \to A_F(\chi)$ is surjective. □

Remark. The lemma holds more generally if K is replaced by any field between K and F.

In the lemma, we can, for example, take χ to be trivial. We find that $A_F(1)$ is trivial. Henceforth, we assume that $\chi \neq 1$. If we take χ to be quadratic of conductor $\ell \equiv 1 \pmod 4$ and let p be odd, we find that $A_F(\chi)$ is isomorphic to the p-Sylow subgroup of the class group of $K = \mathbf{Q}(\sqrt{\ell})$.

Proposition 5. *Let*

$$V = Ker(A_F \to A_{\mathbf{Q}(\zeta_\ell)^+}).$$

If $\chi \neq 1$, then

$$V(\chi) \simeq H^1\big(\mathrm{Gal}\big(\mathbf{Q}(\zeta_\ell)^+/F\big), E_{\mathbf{Q}(\zeta_\ell)^+}\big)(\chi).$$

Proof. Let $\mathfrak{p}$ be a prime of F and let $\mathfrak{q}$ be a prime of $\mathbf{Q}(\zeta_\ell)^+$ above $\mathfrak{p}$. The exact sequence $0 \to U_{\mathbf{Q}(\zeta_\ell)^+_{\mathfrak{q}}} \to (\mathbf{Q}(\zeta_\ell)^+_{\mathfrak{q}})^\times \to \mathbf{Z} \to 0$ yields the exact sequence

$$K_{\mathfrak{p}}^\times \to \mathbf{Z} \to H^1(\pi, U_{\mathbf{Q}(\zeta_\ell)^+_{\mathfrak{q}}}) \to 0.$$

Since the image of the valuation map on $K_{\mathfrak{p}}^\times$ is $e\mathbf{Z}$, we see that

$$H^1(\pi, U_{\mathbf{Q}(\zeta_\ell)^+_{\mathfrak{q}}}) \simeq \mathbf{Z}/e\mathbf{Z},$$

where e is the ramification index of $\mathfrak{q}$ over $\mathfrak{p}$, and the action of Δ on this cohomology group is trivial.

We know from the corollary to Proposition 1 that V is given by locally trivial cohomology classes. Since $\mathbf{Q}(\zeta_\ell)^+/F$ is ramified only at ℓ, and the ramification index there is p^n, we have

$$V \simeq Ker\big(H^1(\pi, E_{\mathbf{Q}(\zeta_\ell)^+}) \to \mathbf{Z}/p^n\mathbf{Z}\big).$$

The result follows. □

Let *Cycl* denote the cyclotomic units of $\mathbf{Q}(\zeta_\ell)^+$, namely, the group generated by elements of the form $(\zeta_\ell^a - \zeta_\ell^{-a})/(\zeta_\ell^b - \zeta_\ell^{-b})$. Let

$$B = E_{\mathbf{Q}(\zeta_\ell)^+}/Cycl.$$

Let I be the augmentation ideal of $\mathbf{Z}[G]$. The exact sequence

$$0 \to I \to \mathbf{Z}[G] \to \mathbf{Z} \to 0$$

implies that there is an isomorphism of Δ-modules

$$\widehat{H}^q(\pi, I) \simeq \widehat{H}^{q-1}(\pi, \mathbf{Z})$$

for all q, where $\widehat{H}$ denotes a Tate cohomology group. Since G is commutative and π and Δ have coprime orders, the group Δ acts trivially on $\widehat{H}^{q-1}(\pi, \mathbf{Z})$ and on $\widehat{H}^q(\pi, \{\pm 1\})$ (see [17, Lemma 1.1]). Therefore, if $\chi \neq 1$,

$$\widehat{H}^q(\pi, I)(\chi) \simeq \widehat{H}^{q-1}(\pi, \mathbf{Z})(\chi) = 0$$

and $\widehat{H}^q(\pi, \{\pm 1\})(\chi) = 0$ for all q. The inverse of the map $I \to Cycl/\{\pm 1\}$ given by

$$\sigma - 1 \mapsto \frac{\sigma(\zeta - \zeta^{-1})}{\zeta - \zeta^{-1}}$$

yields an exact sequence (see [19, Proposition 8.11])

$$0 \to \{\pm 1\} \to Cycl \to I \to 0.$$

This implies that $\widehat{H}^1(\pi, Cycl)(\chi) = 0$ for all q. It follows, when $\chi \neq 1$, that

$$V(\chi) \simeq H^1(\pi, E_{\mathbf{Q}(\zeta_\ell)^+})(\chi) \simeq H^1(\pi, B)(\chi).$$

For a finite Galois module M, let $M^d = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ be the dual of M. The Galois action is given by $(\sigma f)(m) = f(\sigma^{-1}m)$, so the pairing between M and M^d induces a nondegenerate pairing between $M(\chi)$ and $M^d(\chi^{-1})$. Duality theory (see [11]) tells us that $H^1(\pi, B)(\chi) = H^1(\pi, B(\chi))$ is dual to $H_1(\pi, B^d(\chi^{-1}))$. This latter group equals $\widehat{H}^{-2}(\pi, B^d(\chi^{-1}))$, which is isomorphic to $\widehat{H}^0(\pi, B^d(\chi^{-1}))$. Therefore,

$$\big|V(\chi)\big| = \big|\widehat{H}^0\big(\pi, B^d\big(\chi^{-1}\big)\big)\big|.$$

Note that $B^d(\chi^{-1})$ is a module over $\mathcal{O}_\chi[\pi] \simeq \mathcal{O}_\chi[[T]]/(\omega_n(T))$. In [18], it is shown how to use cyclotomic units to compute an ideal $I_{\chi^{-1}} \subseteq \mathcal{O}_\chi[[T]]$ such that

$$B^d\big(\chi^{-1}\big) \simeq \mathcal{O}_\chi[[T]]/I_{\chi^{-1}}.$$

Theorem 1. *Let $\chi \neq 1$. Then $V(\chi)$ is dual to*

$$\big\{f \in \mathcal{O}_\chi[[T]] \;\big|\; Tf \in I_{\chi^{-1}}\big\}/\big(I_{\chi^{-1}} + \big(\omega_n(T)/T\big)\big).$$

Proof. Since $1 + T$ is a generator of π, an element $f \in B^d(\chi^{-1})$ is fixed by π if and only if $Tf = 0$. The norm for π is given by $\omega_n(T)/T$. Since $\widehat{H}^0$ is given by fixed elements modulo norms, the result follows. □

Corollary. *All of $A_K \simeq A_F(\chi)$ capitulates in $\mathbf{Q}(\zeta_\ell)^+$ if and only if $\omega_n(T)/T \in I_{\chi^{-1}}$.*

Proof. Let B_F be the units of F modulo the cyclotomic units of F. It is known (see [14]) that $|A_F(\chi)| = |B_F(\chi)|$. But $B_F(\chi)$ is isomorphic to the π-invariant subgroup of $B(\chi)$ (see [18, Proposition 5.1(i)]). Let σ generate π. Under the pairing between $B(\chi)$ and $B^d(\chi^{-1})$, the annihilator of $(1-\sigma)B^d(\chi^{-1})$ is the π-invariant subgroup of $B(\chi)$. Therefore, $B_F(\chi)$ is dual to, and therefore has the same order as, $B^d(\chi^{-1})/(1-\sigma)B^d(\chi^{-1})$, which is the maximal quotient of $B^d(\chi^{-1})$ on which π acts trivially. This is isomorphic to $\mathcal{O}_\chi[[T]]/(I_{\chi^{-1}} + (T))$, which has the same order as $\{f \in \mathcal{O}_\chi[[T]] \mid Tf \in I_{\chi^{-1}}\}/I_{\chi^{-1}}$ (proof: since $\mathcal{O}_\chi[[T]]/I_{\chi^{-1}}$ is finite, the kernel and cokernel of multiplication by T have the same order). Therefore, the order of $A_F(\chi)$ equals the order of $V(\chi)$ if and only if $\omega_n(T)/T \in I_{\chi^{-1}}$. □

The ideals $I_{\chi^{-1}}$ for small p and for $\ell < 10\,000$ are listed in [18, Tables 4.3 and 4.4]. We give three examples of how to apply Theorem 1.

Example 1. $l = 2089$, $p = 3$, and χ quadratic. In this case π has order 9 and the ring O_χ is equal to $\mathbf{Z}_3$. In [18, Table 4.4], we find that $I_{\chi^{-1}} = (T - 3, 27)$. This implies that the 3-part of the class group of the quadratic field has order 3, namely, the order of $\mathcal{O}_\chi[[T]]/(I_{\chi^{-1}} + (T))$. We have that $T \cdot f \in I_{\chi^{-1}}$ if and only if 9 divides $f(3)$. These power series form an ideal of index 9 in $\mathbf{Z}_3[[T]]$. On the other hand, $\omega_2(T)/T$ is congruent to $(4^9 - 1)/3 = 9 \cdot 3059$ modulo $I_{\chi^{-1}}$, so that the ideal $I_{\chi^{-1}} + (\omega_2(T)/T)$ has index 9 as well. It follows that the module in Theorem 1 is *trivial*. Therefore there is no capitulation.

Example 2. $l = 7489$, $p = 2$ and χ is cubic. In this case π has order 32 and the ring O_χ is the ring $\mathbf{Z}_2[\zeta]$ where ζ denotes a cube root of unity. By [18, Table 4.4], the ideal $I_{\chi^{-1}}$ is equal to $(T+2+4\zeta, 8)$. This implies that the 2-part of the class group of the cubic field has order 4. Since

$$\omega_5(T)/T \equiv \big((-1-4\zeta)^{32} - 1\big)/(-2-4\zeta) \equiv 0 \pmod{I_{\chi^{-1}}},$$

all classes capitulate.

Example 3. $l = 9337$, $p = 2$ and χ is cubic. In this case π has order 4 and the ring O_χ is as in the previous example. By [18, Table 4.4], the ideal $I_{\chi^{-1}}$ is equal to $(T+4-2\zeta, 8)$. Once again the 2-part of the class group of the cubic field has order 4. We have that $T \cdot f \in I_{\chi^{-1}}$ if and only if 4 divides $f(2\zeta)$. These power series form an ideal of index 16 in $O_\chi[[T]]$. On the other hand, $\omega_2(T)/T$ is congruent to $((-3+2\zeta)^4 - 1)/(-4+2\zeta) \equiv 4\zeta \pmod{I_{\chi^{-1}}}$. Therefore the ideal $I_{\chi^{-1}} + (\omega_2(T)/T)$ also has index 16, the module of Theorem 1 is trivial, and there is no capitulation.

6. Numerical data

6.1. Imaginary quadratic fields

Consider imaginary quadratic fields $\mathbf{Q}(\sqrt{-d})$, where $-d$ is the discriminant. There are 31 examples with $d < 100$. Of these, 8 have trivial class groups, 14 have non-trivial class groups that completely capitulate, and the remaining 9 fields ($d = 23, 31, 47, 59, 71, 79, 83, 87, 95$) have elements in their class groups that have orders not dividing 4 and hence do not capitulate. As pointed out above, for sufficiently large d there will always be some elements of the class group that do not capitulate. In fact, most elements will not capitulate.

6.2. Real quadratic fields; $\ell = 3$

In Section 4, we saw that capitulation was fairly predictable for imaginary quadratic fields. It was restricted to the 4-torsion of the class group, and a lot of it could be explained by genus theory. The situation for real quadratic fields seems to be entirely different.

The ideal classes of order 2 capitulate in the cyclotomic field, as we showed in Section 4. However, Proposition 4 gives us no additional information. In fact, it seems that there is no easy way to predict when capitulation occurs.

Consider the 3-parts of the class groups for fields $K = \mathbf{Q}(\sqrt{\ell})$, where $\ell \equiv 1 \pmod 4$ is prime.

There are 52 primes $\ell < 10\,000$ with $\ell \equiv 5 \pmod{12}$ and such that the 3-class group of $\mathbf{Q}(\sqrt{\ell})$ is non-trivial. Since $3 \nmid \phi(\ell)/2$, the ideal classes of order 3 do not capitulate.

On the other hand, when $\ell \equiv 1 \pmod{12}$, the ideal classes of order 3 (and sometimes those of orders 9, 27, ...) have potential capitulation.

Here are the results of some computations.

$\mathbf{Q}(\sqrt{\ell})$, $\ell \equiv 1 \pmod{12}$, $\ell < 10\,000$	
32:	non-trivial 3-class group
26:	maximal 3-capitulation
6:	no 3-capitulation

There seems to be no reason to expect that these are small exceptional cases. In fact, a suitable extension of the Cohen–Lenstra heuristics to include the class groups of a field and a subfield possibly would predict that certain cases of capitulation (covered by part (i) of Lemma 4) and of non-capitulation (covered by part (ii) of Lemma 4) occur with positive density (however, we have not tried to formulate such an extension of the heuristics)

A reasonable prediction from the data is that capitulation is fairly common, and probably fairly random, for cases of potential capitulation.

Here are the details of the computations. They were carried out in PARI.

Part (i) of Lemma 4 shows that there is maximal 3-power capitulation for the following primes:

229, 733, 1129, 1489, 2557, 2677, 2713, 2857, 2917, 3877,
3889, 4597, 4729, 5521, 5821, 6133, 6997, 7057, 7537, 7573,
7753, 8713, 9133.

All of the capitulation in these examples occurs in the sextic subfield of $\mathbf{Q}(\zeta_\ell)$.

Part (ii) of Lemma 4 shows that there is no 3-power capitulation for the following primes:

3229, 5281, 6637, 8017, 8581.

In each of these examples, the 3-part of the class group of the sextic field L is cyclic and $3 \nmid [\mathbf{Q}(\zeta_\ell) : L]$, so no capitulation can occur from L to $\mathbf{Q}(\zeta_\ell)$.

There are four primes remaining: 2089, 4933, 7873, 8761. They can be treated by the methods of Section 5. We find the following:

2089: The class group of K is $\mathbf{Z}/3\mathbf{Z}$. There is no capitulation.

4933: The class group of K is $\mathbf{Z}/3\mathbf{Z}$. It capitulates in the cyclotomic field.

7873: The class group of K is $\mathbf{Z}/9\mathbf{Z}$. The capitulation kernel has order 3, which is maximal capitulation.

8761: The class group of K is $\mathbf{Z}/27\mathbf{Z}$. The capitulation kernel has order 3, which is maximal capitulation.

In all of the above examples, the class group of K is cyclic, and the capitulation, when it occurs, is maximal. For a non-cyclic class group, consider $\ell = 114889$. The class group is $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$. A calculation with PARI shows that the capitulation in the sextic field, hence in $\mathbf{Q}(\zeta_\ell)$, has order 3. Therefore only part of the potential capitulation is actual capitulation in this case.

The majority of the above cases have $\ell \equiv 1 \pmod{12}$ and $3 \| h_K$. When 3 exactly divides the class number h_6 of the sextic subfield of $\mathbf{Q}(\zeta_\ell)$, we are guaranteed to have capitulation. Computations for $\ell < 500\,000$ yield the following:

$K = \mathbf{Q}(\sqrt{\ell})$, $\ell \equiv 1 \pmod{12}$, and $\ell < 500\,000$			
Condition on class number:	$3 \mid h_K$	$3 \| h_K$	$3 \| h_6$
Number of fields:	1343	1181	787

This agrees with the philosophy stated earlier as to why capitulation is common for totally real fields.

6.3. Cyclic cubic fields; $\ell = 2$

The situation for cyclic cubic fields is similar to that for real quadratic fields. Potential capitulation is very often actual capitulation, except for an obstruction that we describe below. We examined the 611 cyclic cubic fields K of prime conductor $\ell < 10\,000$. Of these cubic fields, 505 have trivial class groups. Of the remaining 106, the most common (61 occurrences) class group is $\mathbf{Z}_2 \times \mathbf{Z}_2$. Since the 2-rank of the class group must be even (see [19, Theorem 10.8]) and the class number is prime to 3 (see [19, Theorem 10.4]), this is the smallest possible non-trivial class group. We therefore start by looking at the 2-primary part of the class group.

There are 69 fields with non-trivial 2-primary part of the class group. All classes of order 2 have potential capitulation since $\ell - 1$ is even. Only three cases considered have classes of order 4. One of these has $\ell \equiv 3 \pmod 4$ and therefore there is no capitulation (see below). The remaining two primes (1777 and 4297) have class groups $\mathbf{Z}_4 \times \mathbf{Z}_4$ and these classes have potential capitulation. All classes capitulate for 1777, and the capitulation is $\mathbf{Z}_2 \times \mathbf{Z}_2$ for 4297.

An interesting phenomenon arises. If $\ell \equiv 3 \pmod 4$, then $\mathbf{Q}(\zeta_\ell)^+/K$ has odd degree, so no class of even order capitulates in this subextension. Moreover, the map from the class group of $\mathbf{Q}(\zeta_\ell)^+$ to that of $\mathbf{Q}(\zeta_\ell)$ is injective (see [19, Theorem 4.14]). Therefore, the 2-part of the class group of K does not capitulate in $\mathbf{Q}(\zeta_\ell)$. Of the 69 fields, 34 have $\ell \equiv 3 \pmod 4$, hence there is no capitulation in the 2-part. An interesting question is whether there is an elliptic curve analogue of this phenomenon.

Of the remaining 35 fields (those with $\ell \equiv 1 \pmod 4$), six have none of the 2-part of the class group capitulate, 28 have the entire 2-part capitulate, and one ($\ell = 4297$) has partial capitulation. The following summarizes the computations:

$[K : \mathbf{Q}] = 3$, $K \subset \mathbf{Q}(\zeta_\ell)$, $\ell < 10\,000$	
69:	non-trivial 2-class group
34:	$\ell \equiv 3 \pmod 4$, therefore no 2-capitulation
28:	$\ell \equiv 1 \pmod 4$, maximal 2-power capitulation
1:	$\ell \equiv 1 \pmod 4$, partial 2-capitulation
6:	$\ell \equiv 1 \pmod 4$, no 2-capitulation

For 23 of the fields with $\ell \equiv 1 \pmod 4$, the 2-parts of the class groups of both the cubic field K and the sextic subfield of $\mathbf{Q}(\zeta_\ell)$ are $\mathbf{Z}_2 \times \mathbf{Z}_2$. Part (i) of Lemma 4 implies that the 2-part of the class group of K capitulates.

The remaining cases are treated by the methods of Section 5.

As in the quadratic case, we extended these calculations to count, for $\ell < 500\,000$ and $\ell \equiv 1 \pmod{12}$, how often $4 \| h_K$ (so the 2-part of the class group is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$) and $4 \| h_L$. In this case, capitulation is guaranteed by Lemma 4(i). This situation accounts for a majority of the cases where capitulation occurs.

$[K : \mathbf{Q}] = 3$, $\ell \equiv 1 \pmod{12}$, $\ell < 500\,000$			
Condition on class number:	$2 \mid h_K$	$4 \| h_K$	$4 \| h_L$
Number of fields:	1447	1328	933

Again, this agrees with the philosophy stated earlier as to why capitulation is common for totally real fields.

There are 24 primes less than 10 000 where the 7-part of the class group is non-trivial. Of these, 21 have $\ell \not\equiv 1 \pmod 7$, therefore do not have potential capitulation. The remaining 3 cases have maximal capitulation. The most interesting case is 7351, which has class group $\mathbf{Z}_{49}$ and can be treated by the method of Section 5.

For the remaining cubic fields K with non-trivial class group, there is no potential capitulation for any ideal classes. This is not surprising since, for example, we expect only 1/18 of the primes with 19 in the class number of K to have $\ell \equiv 1 \pmod{19}$, and there are not enough examples (7 in the case of 19) for this to be very likely.

6.4. Real quadratic fields; $\ell = 5$

There are 259 primes $\ell < 500\,000$ such that $\ell \equiv 1 \pmod{20}$ and the class number of $\mathbf{Q}(\sqrt{\ell})$ is a divisible by 5. We have the following data:

$K = \mathbf{Q}(\sqrt{\ell})$, $\ell \equiv 1 \pmod{20}$, $\ell < 500\,000$	
259:	non-trivial 5-class group
227:	maximal 5-capitulation
32:	no capitulation

For $\ell < 400\,000$, the quadratic fields with non-trivial 5-class group were found, and then the class group of the degree 10 subfield of $\mathbf{Q}(\zeta_\ell)$ was computed using PARI. In these 203 cases, there are 168 where the class group of both the quadratic field and the tenth degree field have 5-class group cyclic of order 5. Therefore, Lemma 4 implies that all classes of order 5 capitulate. Moreover, of the 203 fields with $\ell < 400\,000$, there are 147 such that the entire class group (not just the 5-part) of the quadratic field is isomorphic to the class group of the tenth degree field. This agrees with the philosophy that high degree totally real fields tend to have small class numbers.

For $400\,000 < \ell < 500\,000$, the class number calculations started taking a long time, so the methods of Section 5 were used to determine the capitulation behavior. They were also used for the smaller prime $\ell = 154\,501$ to determine that its 5-class group capitulates in $\mathbf{Q}(\zeta_{154\,501})$. These methods are much faster than computing the full class number. This is to be expected, since we need only the 5-part of the class number (however, calculating the 5-part of the units modulo cyclotomic units could be used to compute the 5-part of the class number quickly, thus yielding an alternative approach for the present purposes).

It seems worth mentioning a few details of computing the tenth degree field. For a prime $\ell \equiv 1 \pmod{10}$, the polynomial of the fifth degree subfield of $\mathbf{Q}(\zeta_\ell)$ can be computed using formulas of Tanner and Lehmer [13]. This yields a degree 5 polynomial $P(X)$. The following lemma shows that $P(X+\sqrt{\ell})P(X-\sqrt{\ell})$, which can be computed numerically to sufficient accuracy and then rounded to a polynomial with integral coefficients, is then the irreducible tenth degree polynomial that gives the desired field. This method works well for small primes, but is slow for large primes ℓ, say $\ell > 400\,000$.

Lemma 6. *Let L/K be a Galois extension of fields such that $\mathrm{Gal}(L/K)$ is cyclic of order mn with $\gcd(m,n) = 1$. Let F_1 be the subfield of degree m over K and let F_2 be the subfield of degree n over K. Write $F_1 = K(\alpha)$ and $F_2 = K(\beta)$. Then $L = K(\alpha+\beta)$.*

Proof. Since $n = [K(\alpha)(\alpha+\beta) : K(\alpha)]$ divides $[K(\alpha+\beta) : K]$, and similarly m divides this degree, the degree is at least $mn = [L : K]$. □

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